

REGGE BEHAVIOUR AND REGGE TRAJECTORY FOR LADDER GRAPHS IN SCALAR ϕ^3 FIELD THEORY

by

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Abstract

Using the gaussian representation for propagators (which can be proved to be exact in the infinite number of loops limit) we are able to derive the Regge behaviour for ladder graphs of ϕ^3 field theory in a completely new way. An analytic expression for the Regge trajectory $\alpha(t/m^2)$ is found in terms of the mean-values of the Feynman α -parameters. $\alpha(t/m^2)$ is calculated in the range $-3.6 < t/m^2 < 0.8$. The intercept $\alpha(0)$ agrees with that obtained from earlier calculations using the Bethe-Salpeter approach for $\alpha(0) \gtrsim 0.3$.

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The first attempts at calculating ladder graphs and their asymptotic behaviour, ($s \rightarrow \infty$, t fixed) i.e. Regge behaviour with leading Regge trajectory $\alpha(t)$, date back to more than thirty years ago. Used were the Feynman α -parameter representation¹, the Bethe-Salpeter^{2,4} equation or the multiperipheral model³ which gives a recursion equation close to the Bethe-Salpeter one. Nakanishi⁴ derived an exact expression for the intercept $\alpha(0)$ of the leading Regge trajectory assuming that a massless scalar was exchanged in the central propagators joining the sides of ladder diagrams. More recently, ladders diagrams in massless ϕ^3 were given an expression in terms of polylogarithms⁵, a leading log approximation⁶ used to obtain the QCD Pomeron trajectory and the asymptotics of ϕ^3 ladders in six dimensions studied for large t ⁷.

Here, we want to exploit the α -parameter representation for the Feynman ladder graphs in massive scalar ϕ^3 theory at $d = 4$ dimensions to obtain, first, the Regge behaviour and, second, the leading Regge trajectory $\alpha(t)$. Our approach, however, will be very different from all those tried before. Some time ago, we gave a proof⁸ that when there are an infinite number of them, the α -parameters can be replaced by their mean-values, once an overall scale has been extracted, and this for superrenormalizable scalar field theories. Let us define a homogeneous polynomial $P_G(\{\alpha\})$

$$P_G(\{\alpha\}) = \sum_{\mathcal{T}} \prod_{\ell \notin \mathcal{T}} \alpha_\ell \quad (1)$$

which is a sum over all spanning trees \mathcal{T} belonging to a one-particle irreducible graph G (a spanning tree is a tree incident with all vertices of G). $P_G(\{\alpha\})$ is of degree L in the α_ℓ 's if G has L loops. Similarly, let us define also

$$Q_G(P, \{\alpha\}) = P_G^{-1}(\{\alpha\}) \sum_C s_C \prod_{\ell \in C} \alpha_\ell \quad (2)$$

where we have summed \sum_C over all cuts C on G (a cut C is the complement of a tree \mathcal{T} on G plus one propagator. This propagator cuts \mathcal{T} in two disjoint parts). Then, in (2)

$\sum_C s_C \prod_{\ell \in C} \alpha_\ell$ is a homogeneous polynomial of degree $L + 1$ in the α_ℓ 's. Then,

$$s_C = \left(\sum_{v \in G_1} P_v \right)^2 = \left(\sum_{v \in G_2} P_v \right)^2 \quad (3)$$

is a Lorentz-invariant quadratic in the external momenta P_v associated with the external lines of G_1 (or G_2) if G_1 (or G_2) is one part of G . $P_G(\{\alpha\})$ and $Q_G(P, \{\alpha\})$ contain all the information about the topology of G . Let us define by F_G the Euclidean amplitude for G , then for $I, L \rightarrow \infty$, (I is the number of propagators of G) coupling $\gamma = -1$, we get

$$F_G = (4\pi)^{-dL/2} h_0 [P_G\{\bar{\alpha}\}]^{-d/2} [Q_G(P, \{\bar{\alpha}\}) + m^2 h_0]^{-(I-d/2L)} \Gamma(I - d/2L) h_0^{I-1} / (I-1)! \quad (4)$$

where $h_0 = \sum_\ell \alpha_\ell$ and $h_0^{I-1} / (I-1)!$ is the phase-space volume of the α_ℓ 's. The factor $\Gamma(I - d/2L) [Q_G(P, \{\bar{\alpha}\}) + m^2 h_0]^{-(I-d/2L)}$ coming from the integration over the overall scale is convergent for superrenormalizable theories (such as ϕ^3 in $d = 4$). The next step will consist in evaluating the mean-values $\bar{\alpha}_\ell$. Then, the result will be put in (4) to obtain the Regge behaviour, taking into account the peculiar form of $P_G(\{\bar{\alpha}\})$ and $Q_G(P, \{\bar{\alpha}\})$ for ladder graphs.

To calculate the mean-value $\bar{\alpha}_i$ of a Feynman parameter α_i attached to a propagator i one has to isolate the dependence of $P_G(\{\alpha\})$ and $Q_G(P, \{\alpha\})$ on this particular α_i . As we deal with polynomials of degree zero or one in each α_ℓ we write

$$P_G(\{\alpha\}) = a_i + b_i \alpha_i = b_i (a_i/b_i + \alpha_i) \quad (5.a)$$

$$\sum_C s_C \prod_{\ell \in C} \alpha_\ell = d_i + e_i \alpha_i = e_i (d_i/e_i + \alpha_i) \quad (5.b)$$

where a_i , b_i , d_i and e_i are polynomials not depending explicitly on α_i . First, one can prove⁹ that a_i/b_i and d_i/e_i are equal in the limit where the number of propagators I is infinite. Then, $Q_G(P, \{\alpha\}) = e_i/b_i$ do not depend explicitly on α_i in this limit. However,

due to the constraint $h_0 = \sum_i \alpha_i$, we have $\sum_{j \neq i} \alpha_j = h_0 - \alpha_i$ and α_i reappears each time h_0 does. And it does so each time a power of $\bar{\alpha}_j$ appears because⁸

$$\bar{\alpha}_\ell = 0(h_0/I) \quad (6)$$

as suggested by remarking that $h_0^{I-1}/(I-1)! \sim (eh_0/I)^I$ leaving a phase space $\sim eh_0/I$ for each α_ℓ . Gathering everything together we get F_G to depend on α_i through the integral⁸

$$F_G = h_0(4\pi)^{-dL/2}(\bar{b}_i)^{-d/2} [Q_G(P, \{\bar{\alpha}\}) + m^2 h_0]^{-(I-d/2L)} \Gamma(I - d/2L).$$

$$[h_0^{I-2}/(I-2)!] \int_0^{h_0} d\alpha (\mu_i + \alpha_i)^{-d/2} \exp\{-(I - d/2L)(1 - \beta)\alpha_i/h_0\} H_i(\alpha_i - \bar{\alpha}_i) \quad (7)$$

where we used the notation $\mu_i = \bar{a}_i/\bar{b}_i$ and $1 - \beta = [1 + Q_G(P, \{\bar{\alpha}\})/(h_0 m^2)]^{-1}$, \bar{a}_i , \bar{b}_i and $\{\bar{\alpha}\}$ being respectively a_i , b_i and $\{\alpha\}$ where α_j is replaced by $\bar{\alpha}_j$ (for all $j \neq i$). $H_i(\alpha_i - \bar{\alpha}_i)$ represents the variation due to the dependence of the mean values $\bar{\alpha}_j$ on α_i (not included in the replacement $h_0 \rightarrow h_0 - \alpha_i$) when α_i is different from $\bar{\alpha}_i$ in $(\bar{b}_i)^{-d/2}[Q_G(P, \{\bar{\alpha}\}) + m^2 h_0]^{-(I-d/2L)}$. This factor which was not included in our first paper⁸ does not invalidate its conclusions concerning the validity of the mean-value approach and the validity of (6). Its computation is, however, rather involved and will be published elsewhere¹⁰. Because of the exponential in (7), we can write

$$\bar{\alpha}_i = \bar{g}(\mu_i, H_i)(h_0/L + 1)[1 + Q_G(P, \{\bar{\alpha}\})/(h_0 m^2)] \quad (8)$$

where $I - d/2L$ has been replaced by $L + 1$ for $d = 4$ and ϕ^3 . Then, $\bar{g}(\mu_i, H_i)$ is a constant to be determined by the equation (7). All what we have said until now are general considerations. We have to specialize those for ladder graphs. In the limit where $L \rightarrow \infty$ we get¹¹

$$P_G(\{\bar{\alpha}\}) = (\bar{\alpha}_-)^L \exp(yL) f(y) \quad (9.a)$$

$$f(y) = \frac{1}{2}y(1 + y^{-1})^2 \quad (9.b)$$

with $y = (2\bar{\alpha}_+/\bar{\alpha}_-)^{1/2}$, $\bar{\alpha}_+$ being the mean-value of the $\bar{\alpha}_\ell$'s for the propagators parallel to the ladder and $\bar{\alpha}_-$ being the mean-value of the $\bar{\alpha}_\ell$'s for the central propagators joining the two sides of the ladder. In the same limit we get¹¹

$$Q_G(P, \{\bar{\alpha}\}) = (t/2) L \bar{\alpha}_+ + s \bar{\alpha}_- \exp(-yL)[f(y)]^{-1} . \quad (10)$$

So, putting the value for $P_G(\{\bar{\alpha}\})$ in (9.a) with $\bar{\alpha}_-$ given by (8) with $\bar{g}(\mu_\ell, H_i) = \bar{g}_-$ we get (putting back also the factor for the coupling constant $(-\gamma)^{2L+2}$) using asymptotic expressions for factorials

$$F_G = (e^2/\sqrt{3})[-\gamma/(mf(y))]^2[-\gamma e/(m4\pi3\sqrt{3})]^{2L}.$$

$$[\exp(-yL)/\bar{g}_-]^{2L}[1 + Q_G(P, \{\bar{\alpha}\})/(h_0 m^2)]^{-(3L+1)} . \quad (11)$$

Now let us sum over L the ladder amplitudes in (11) and find a saddle-point. Then, the saddle point equation reads

$$2 \ell n C^{st} + 3 \ell n(1 - \beta) + (3L + 1) \left[y + \frac{1}{L+1} \right] bs/(1 - \beta) = 0 \quad (12)$$

with :

$$C^{st} = [-\gamma e/(m4\pi3\sqrt{3})][\exp(-y/\bar{g}_-)] \quad (13.a)$$

$$\beta = a + bs \quad (13.b)$$

$$a = (t\bar{g}_-/(2m^2))(\bar{\alpha}_+/\bar{\alpha}_-) \quad (13.c)$$

$$bs = [\bar{g}_-/(m^2 f(y))] s \exp(-yL)/(L+1) . \quad (13.d)$$

We readily note that as $s \rightarrow \infty$, no solution exists for finite L . This is consistent with the $L \rightarrow \infty$ limit we have used until now.

In the same way bs tending to a constant yields no solution. If $bs \rightarrow \infty$ we get F_G to explode like $(-bs)^L$. The only possibility left is bs tending to zero. In that case β tends to a and we get

$$(1 - a)[(2/3)\ell n C^{st} + \ell n(1 - a)] + L y bs = 0 \quad (14)$$

yielding the condition

$$s \exp(-yL_{sp}) = s_0 \quad , \quad s_0 \text{ constant} \quad (15.a)$$

with

$$-(1-a)[(2/3)\ell n C^{st} + \ell n(1-a)] = s_0 y \bar{g}_-/(m^2 f(y)) \quad . \quad (15.b)$$

Thus, the saddle-point value for L , $L_{sp} = y^{-1}\ell n(s/s_0)$ which is *a very natural condition* because it means that the dominant ladders have a length which is proportional to the “rapidity” $\ell n(s/s_0)$ which has a very important role in high energy physics. Replacing L by L_{sp} in (11) gives the leading Regge trajectory immediately

$$\alpha(t/m^2) = y^{-1}[2 \ell n C^{st} + 3 \ell n(1-\beta)] \quad . \quad (16)$$

The trajectory $\alpha(0)$ can be computed if we know $\bar{\alpha}_+$ and $\bar{\alpha}_-$. In fact, we do not compute $\bar{\alpha}_+$ and $\bar{\alpha}_-$ themselves which are $0(1/L)$, (see (8)), but the finite quantities

$$\bar{\alpha}_- \Lambda = \bar{g}_-(\mu_-, H_i) \quad (17.a)$$

$$\bar{\alpha}_+ \Lambda = \bar{g}_+(\mu_+, H_i) \quad (17.b)$$

with the scale Λ defined by

$$\Lambda = [(L+1)/h_0](1-\beta) \quad . \quad (18)$$

Then, $y = (2\bar{\alpha}_+ \Lambda / (\bar{\alpha}_- \Lambda))^{1/2}$ is also known. By inspection (13) also shows that β is known once $\bar{\alpha}_- \Lambda$ and $\bar{\alpha}_+ \Lambda$ are. The fact that F_G is expressed in two ways in (4) and (7) allows to have an equation for each $\bar{\alpha}_i$. Here, we have only two different $\bar{\alpha}_i$ ’s, $\bar{\alpha}_+$ and $\bar{\alpha}_-$ which give a total of two coupled equations. For a given $\bar{\alpha}_i$ we have the equation

$$\exp(-(2+\beta)/3)[I/(h_0\Lambda)] \int_0^\infty d(\alpha_i\Lambda) [(\mu_i\Lambda + \bar{\alpha}_i\Lambda)/(\mu_i\Lambda + \alpha_i\Lambda)]^{-d/2} \cdot$$

$$\exp(-\alpha_i\Lambda) H_i(\bar{\alpha}_i - \alpha_i) = 1 \quad (19)$$

In the particular case of ladders μ_+ and μ_- are known functions of $\bar{\alpha}_+$ and $\bar{\alpha}_-$ through¹¹ $1 + \bar{\alpha}_-/\mu_- = y^{-1}$ and $1 + \mu_+/\bar{\alpha}_+ = 2y^{-1}$. The factor $\exp(-(2 + \beta)/3)$ takes into account the fact that in (4) $\bar{\alpha}_i = 0(h_0/I)$ whereas in (7) $\bar{\alpha}_i = 0(h_0/(I-1))$ because the mean-values are taken with only $(I-1)$ variables instead of I in (4). A more careful examination of the equations (19) (and in particular how to determine $H_i(\bar{\alpha}_i - \bar{\alpha}_c)$) will be published elsewhere¹⁰.

We can now look at the solution for $\alpha(t)$ given in terms of $\ell n \gamma_m$ where $\gamma_m = \gamma e/(m4\pi3\sqrt{3})$ (in Minkowski space where $\gamma > 0$).

In fig. 1 we displayed the Regge trajectory $\alpha(t/m^2)$ (in Minkowski space, t being multiplied by -1 relative to the Euclidean space) for two values of $\ell n \gamma_m$, 0 (upper curves) and -0.1 (lower curves). Due to the imperfect knowledge of $H_i(\alpha_i - \bar{\alpha}_i)$ which can be represented in at least two different ways we (at present) have an uncertainty which is shown by the magnitude of the difference between the continuous curve and the dotted curve (for a given γ_m). The squares show the values of the intercept $\alpha(0)$ as can be calculated from a Bethe-Salpeter calculation^{4,12} assuming $m = 0$ for the central rungs of the ladders. We show the results obtained for $-3.6 \leq t/m^2 \leq 0.8$. For t/m^2 lower than -4 we have problems with our algorithm solving (19). For t/m^2 larger than 0.8 we have a big uncertainty which is correlated with, we think, the solution becoming complex well below the expected threshold at $t = 4m^2$. Maybe a new type of anomalous threshold (of saddle point type ?¹³) is showing up or, even simpler, our method plainly fails to give meaningful results for large positive t/m^2 . We note that the shapes obtained are similar to those calculated¹⁴ for the Pomeron using the Lipatov equation. In fig. 2 the intercept $\alpha(0)$ is displayed as function of γ_m for $-0.5 < \ell n \gamma_m < 0.4$. Again the uncertainty is shown by two curves, plain and dotted.

We have a good agreement for $\alpha(0) \gtrsim 0.3$ with the Bethe-Salpeter calculation^{4,12}. For lower values of $\alpha(0)$ γ_m becomes small and the contribution outside the saddle-point starts

to dominate. In particular we know that for $\gamma_m \rightarrow 0$, the diagrams with a finite number of rungs will dominate the contribution to the amplitude, giving an intercept tending to - 1. For large values of $\ell n \gamma_m$, again, we encounter larger errors and so we didn't display the results there. We hope to extend these calculations in the framework of QCD.

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Figure Captions

Fig. 1 The Regge trajectory $\alpha(t/m^2)$ as displayed for two values of $\ell n \gamma_m$, 0. for the upper curves and - 0.1 for the lower curves. The size of the uncertainty is given by the difference between the corresponding plain and dotted curves. Squares indicate the intercept $\alpha(0)$ obtained from the Bethe-Salpeter equation^{4,12}.

Fig. 2 The intercept $\alpha(0)$ is shown for the range $-.5 \leq \ell n \gamma_m \leq .4$. Again, the uncertainty is given by the difference between the plain and the dotted curves. The Bethe-Salpeter result is also displayed with the dashed line supporting empty squares.

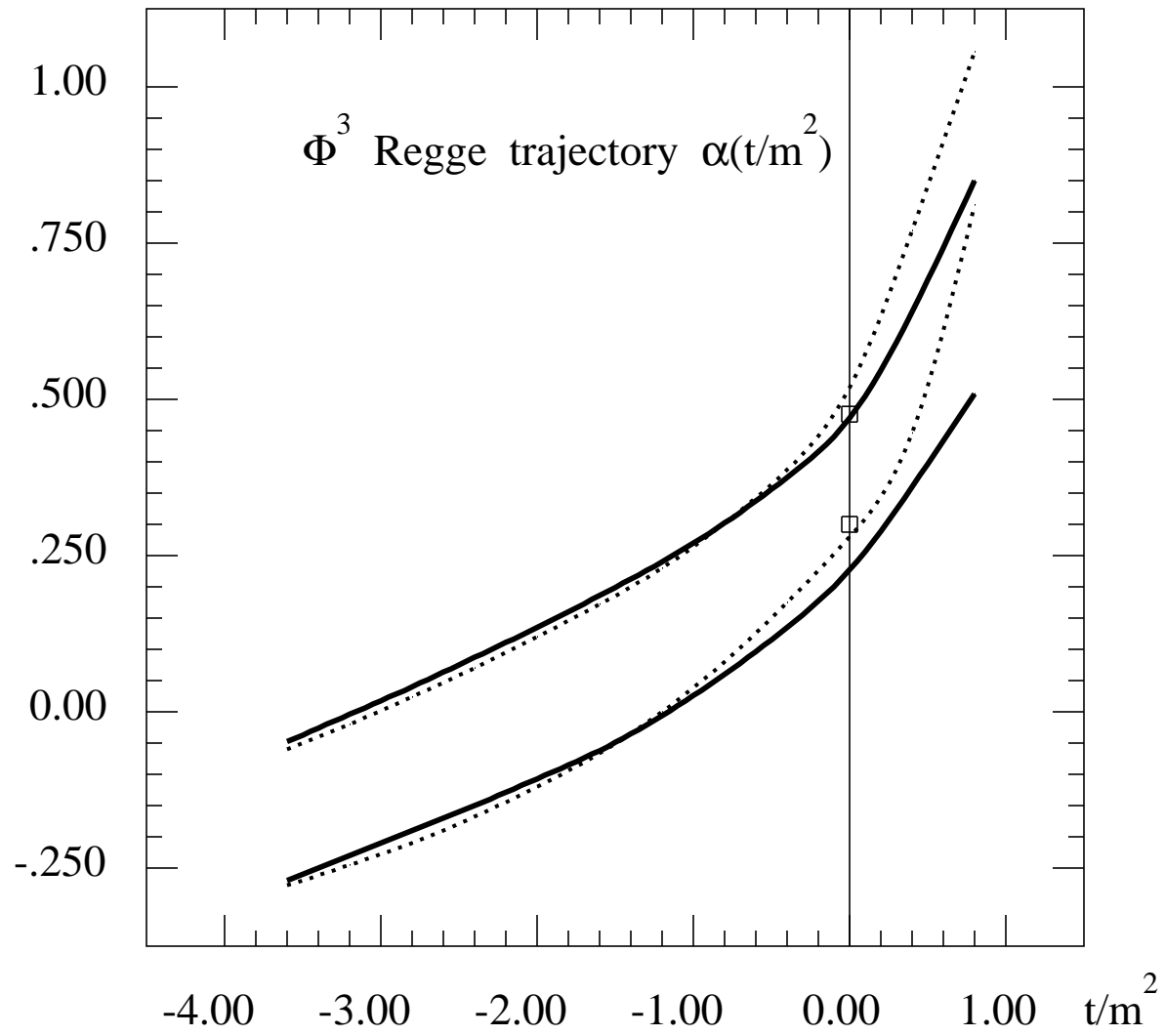


Fig. 1

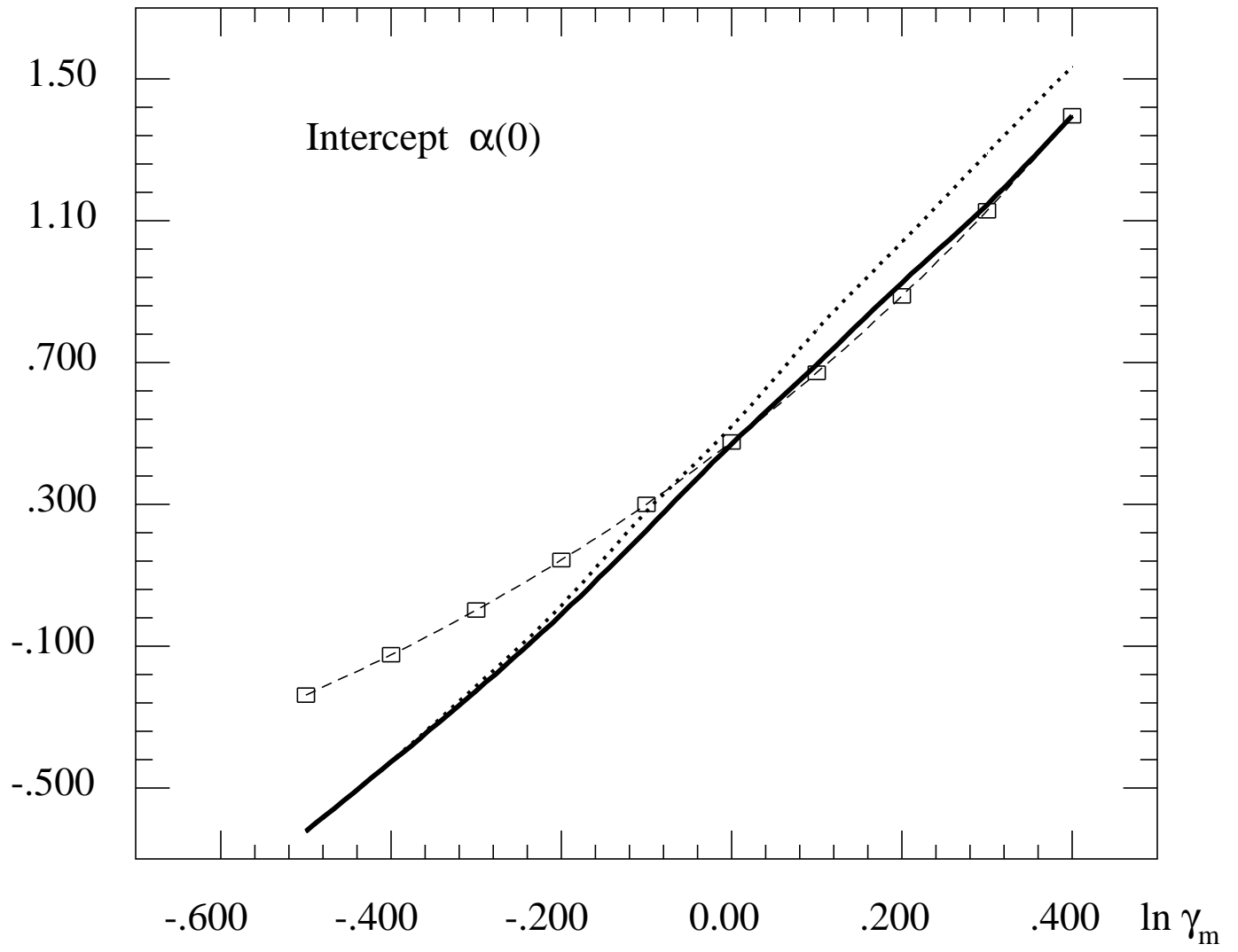


Fig. 2